

Instabilities of a birefringent semi-metal

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Birefringent fermions arise as massless fermionic low energy excitations of a particular tight binding model for spinless fermions on a square lattice which have two “speeds of light” [M. P. Kennett, *et al.*, Phys. Rev. A **83**, 053636 (2011)]. We use mean field theory to study phases that can arise when there are nearest neighbour and next-nearest neighbour repulsive interactions in this model and demonstrate robustness of the birefringent semi-metal phase in the presence of weak interactions and identify transitions to staggered density and quantum anomalous Hall ordered phases. We consider the effect of coupling birefringent fermions to a magnetic field, and find analytic expressions for the corresponding Landau levels and demonstrate that their integer Quantum Hall effect displays additional plateaux beyond those observed for regular Dirac fermions, such as in graphene. We briefly discuss a tight-binding construction that leads to three dimensional birefringent fermions.

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I. INTRODUCTION

There has been recent intense experimental and theoretical activity focused on systems with low energy excitations with Dirac dispersions, such as Graphene,¹ topological insulators,² and Weyl semi-metals.³ Birefringent fermions are massless fermions which differ from Dirac fermions in that they have more than one distinct velocity. It has recently been shown that they can arise as the low energy excitations of a specific tight binding model⁴ and are one of a class of recently investigated birefringent Dirac systems, in which there may be multiple Fermi velocities and/or flat bands.^{4–16} The most promising venue for realizing such physics appears to be using cold atoms in optical lattices.^{4–7,9,10,13}

The recent demonstration of artificial Dirac systems, in cold atoms,¹⁷ “molecular graphene”¹⁸ and dielectric resonators¹⁹ opens the door to engineering Dirac-like bandstructures and exploring their properties. This motivates our study of birefringent fermions as an example of a system that generalizes regular Dirac fermions. These fermions break the chiral $SU(2)$ symmetry present for Dirac fermions,²⁰ but do so without generating a mass,⁴ unlike the usual case for Dirac fermions.²¹ The price that is paid is that the emergent low energy Lorentz symmetry is also broken, and so one has a situation where there are two different Fermi velocities (or “speeds of light”). In detail, the low energy theory of birefringent fermions consists of four component massless fermions with two separate Fermi velocities $v_0(1 \pm \beta)$ controlled by the parameter $0 \leq \beta \leq 1$. Writing the low energy theory in Dirac form, the parameter β multiplies terms in the kinetic energy not present in the regular Dirac Hamiltonian. We have considered the response of these fermions to a variety of perturbations, and in the presence of topological defects^{4,15} and found that the property of birefringence is quite robust. An important question to ask is whether this birefringence is robust in the presence of interactions, and the nature of broken symmetry phases that gap the birefringent semi-metal for strong enough interactions.

We consider this question here by treating repulsive interactions at a mean-field level in the previously introduced tight-binding model of spinless fermions which has birefrin-

gent fermions as its low energy excitations.⁴ The tight binding model we consider in fact interpolates between a model of regular Dirac fermions on a square lattice and the Lieb lattice, which has attracted considerable attention itself recently.^{22–24} We find that generically the birefringent semi-metal phase is stable to weak interactions. For sufficiently strong nearest neighbour interactions there is an instability to a staggered density phase and this tendency is enhanced as birefringence increases in strength. For sufficiently strong next-nearest neighbour interactions, there can be a topologically insulating quantum anomalous Hall phase, which is robust to weak nearest neighbour interactions.

This paper is structured as follows. In Sec. II we recall the model of birefringent Dirac fermions. In Sec. III we study the phases that arise due to both nearest neighbour and next-nearest neighbour interactions. In Sec. IV and Sec. V we consider the effects of magnetic field and the generalization of birefringent fermions to three dimensions respectively. Finally, in Sec. VI we conclude and discuss our results.

II. BIREFRINGENT FERMIONS

We recently introduced birefringent fermions as the low energy excitations of the tight binding model of non-interacting spinless fermions on a square lattice at half-filling illustrated in Fig. 1.⁴ This tight-binding model can also be viewed as corresponding to a model with positive hopping parameters and half a flux quantum through each plaquette, similar to a square lattice model considered by Seradjeh *et al.*²⁵ which admits Dirac fermions as low energy excitations.

The dispersion relation reads as

$$E_k = \pm 2J_{\pm} \sqrt{\cos^2 k_x + \cos^2 k_y}, \quad (1)$$

(the factor of 2 in Eq. (1) corrects Ref. 4) where $J_{\pm} = J_0(1 \pm \beta)$, with $0 \leq \beta \leq 1$. This leads to four equivalent Dirac points at the corners of the Brillouin zone: $\mathbf{K}_{\pm, \pm} = (\pm \frac{\pi}{2}, \pm \frac{\pi}{2})$. Labelling the four sites in the unit cell as A, B, C , and D we

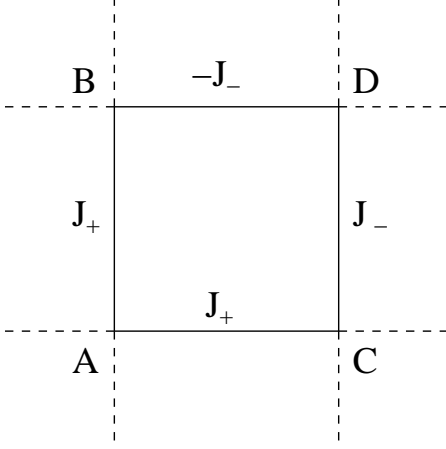


FIG. 1. Unit cell of tight binding model

can write the low energy theory in the form:

$$H = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger [E_{\mathbf{k}} - H_{\mathbf{k}}] \psi_{\mathbf{k}}, \quad (2)$$

where $\psi_{\mathbf{k}}^T = (c_{A\mathbf{k}}, c_{B\mathbf{k}}, c_{C\mathbf{k}}, c_{D\mathbf{k}})$, with $c_{I\mathbf{k}}$ a fermionic annihilation operator for a fermion with momentum \mathbf{k} which resides on sites $I = A, B, C$, or D , and (setting $2J_0 = 1$)

$$H_{\mathbf{k}} = [i(\gamma_0\gamma_1 + i\beta\gamma_3)k_x + i(\gamma_0\gamma_2 + i\beta\gamma_5)k_y] \quad (3)$$

$$= H_0 + H_\beta,$$

where H_0 is the Hamiltonian when $\beta = 0$ and H_β contains all terms involving β . We use a non-standard representation of the gamma matrices in which $\gamma_0 = \sigma_3 \otimes \sigma_3$, $\gamma_1 = \sigma_2 \otimes I_2$, $\gamma_2 = \sigma_3 \otimes \sigma_2$, $\gamma_3 = -\sigma_1 \otimes I_2$, and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -\sigma_3 \otimes \sigma_1$. The matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ and γ_5 satisfy the Clifford algebra $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$.²⁶ The representation is four dimensional, the minimal dimension for a time-reversal invariant system of spinless Dirac fermions in two dimensions on a lattice.²⁷ There are four Dirac points, however unlike graphene, for which the minimal representation is constructed with two sublattice degrees of freedom and two inequivalent Dirac points, in the birefringent model the four Dirac points are equivalent and arise from there being four lattice points in the unit cell. The spectrum

$$E_{\mathbf{k}} = \pm(1 \pm \beta)|\mathbf{k}|,$$

can be obtained very simply by noting that H_0 and H_β commute and that both H_0 and H_β individually represent Dirac Hamiltonians (albeit for different representations of the gamma matrices). The two Dirac cones with Fermi velocities $1 \pm \beta$ make it tempting to think that it might be possible to use a direct sum with two copies of two component massless Dirac fermions or Weyl fermions. This is not possible without breaking time reversal symmetry,^{28,29} which is respected here.²⁷ In the limit $\beta = 1$ the model is analogous to the previously studied case of the Lieb lattice^{6,7,30} and there are two flat bands at zero energy and a two component Weyl fermion.

III. INTERACTIONS

We focus on nearest neighbour and next-nearest neighbour repulsive interactions introduced via the Hamiltonian

$$H_{\text{int}} = \sum_{i,j} V_{ij} \hat{n}_i \hat{n}_j, \quad (4)$$

where we note that for spinless fermions there can be no on-site interactions. We study the low energy theory in the vicinity of one of the Dirac cones at the corners of the Brillouin zone and ignore scatterings between different Dirac cones. We represent the generating functional as an imaginary time path integral over Grassmann-valued fields ψ and $\bar{\psi} = \psi^\dagger \gamma_0$:

$$\mathcal{Z} = \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi] e^{-S[\bar{\psi},\psi]}, \quad (5)$$

where $S[\bar{\psi}, \psi] = \int_0^\beta d\tau \mathcal{L}(\bar{\psi}, \psi)$ is the action and the Lagrangian \mathcal{L} is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\beta + \mathcal{L}_{\text{int}}, \quad (6)$$

with \mathcal{L}_0 the Lagrangian associated with H_0 , \mathcal{L}_β the Lagrangian associated with H_β and \mathcal{L}_{int} the interaction Lagrangian. We treat the interactions at a mean field level, similarly to approaches previously used for graphene^{20,31} by solving the saddle point equations for the order parameters obtained from the path integral formalism.

A. Nearest Neighbour interactions

We first consider nearest neighbour interactions with strength V_1 . We can decouple the quartic interaction terms in the action by introducing Hubbard-Stratonovich fields and making use of the identities corresponding to Hartree and Fock decompositions of

$$n_A n_B + n_A n_C + n_B n_D + n_C n_D,$$

which are written out explicitly in Appendix A 1.

In principle we should introduce Hubbard-Stratonovich fields corresponding to all Hartree and Fock decompositions of the interaction term, but we find that for nearest neighbour interactions the leading instability is to staggered density order with order parameter

$$\langle \chi \rangle \propto -\langle \bar{\psi} \psi \rangle = \langle n_B \rangle + \langle n_C \rangle - \langle n_A \rangle - \langle n_D \rangle.$$

Keeping only the Hartree term [Eq. (A1)] and also introducing the field

$$\phi \propto n_A + n_B + n_C + n_D,$$

we get

$$\mathcal{Z} = \int [\mathcal{D}\chi][\mathcal{D}\phi] e^{-S[\chi] - S[\phi]} \int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi] e^{-S_0[\bar{\psi},\psi] - S_{\text{mix}}}, \quad (7)$$

where

$$\begin{aligned} S[\chi] &= \frac{1}{2V_1} \int_0^\beta d\tau \int d^2\vec{x} [\chi(\vec{x}, \tau)]^2, \\ S[\phi] &= \frac{1}{2V_1} \int_0^\beta d\tau \int d^2\vec{x} [\phi(\vec{x}, \tau)]^2, \\ S_0[\bar{\psi}, \psi] &= \int_0^\beta d\tau [\mathcal{L}_0 + \mathcal{L}_\beta[\bar{\psi}, \psi]], \end{aligned}$$

and

$$S_{\text{mix}} = \int_0^\beta d\tau \int d^2\vec{x} [\chi\bar{\psi}\psi + i\phi\bar{\psi}\gamma_0\psi].$$

After integrating out the Grassman fields, we may write the

generating functional as

$$\mathcal{Z} = \int [\mathcal{D}\chi][\mathcal{D}\phi] e^{-S[\chi] - S[\phi] + \text{Tr}(\ln M)},$$

where after Fourier transforming,

$$M = i\gamma_\mu k_\mu - \beta\gamma_0\gamma_3 k_1 - \beta\gamma_0\gamma_5 k_2 + \chi + i\phi\gamma_0.$$

We take a saddle point approximation:

$$\frac{\langle\chi\rangle}{V_1} = \text{Tr}[M^{-1}], \quad \frac{\langle\phi\rangle}{V_1} = \text{Tr}(i\gamma_0 M^{-1}),$$

and find that the saddle point equation for χ (where Λ is an ultra-violet cutoff) gives the critical interaction strength, V_c as

$$\frac{1}{V_c} = \frac{1}{2\pi^2} \int_{-1}^1 dx \int_0^\Lambda dk \left\{ \frac{k^2}{\chi^2 + k^2[(1-\beta)^2 + x^2\beta(2-\beta)]} + \frac{k^2}{\chi^2 + k^2[(1+\beta)^2 - x^2\beta(2+\beta)]} \right\}. \quad (8)$$

We find that $\phi = 0$ and that a non-zero solution for χ may be found provided $V_1 \geq V_c$, where (keeping only terms that scale with Λ) we can obtain V_c as a function of β :

$$\frac{1}{V_c} = \frac{\Lambda}{\pi^2} \left\{ \frac{1}{(1-\beta)\sqrt{\beta(2-\beta)}} \tan^{-1} \left(\frac{\sqrt{\beta(2-\beta)}}{1-\beta} \right) + \frac{1}{2(1+\beta)\sqrt{\beta(2+\beta)}} \ln \left| \frac{1 + \frac{1+\beta}{\sqrt{\beta(2+\beta)}}}{1 - \frac{1+\beta}{\sqrt{\beta(2+\beta)}}} \right| \right\}, \quad (9)$$

which is illustrated in Fig. 2. The result reduces to the previously calculated expression in the limit $\beta \rightarrow 0$.³²

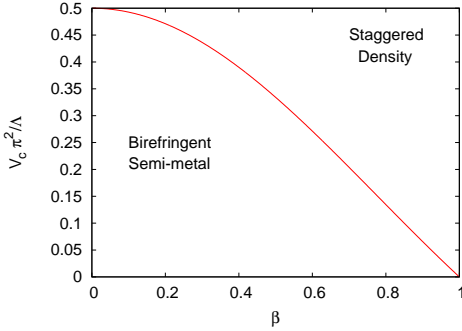


FIG. 2. Phase diagram in the presence of nearest neighbour interactions as a function of V_1 and β .

Figure 2 illustrates that as β approaches unity, the birefringent semi-metal becomes increasingly more susceptible to staggered density order. In the presence of a non-zero staggered density order parameter, χ , the spectrum of the birefringent fermion model becomes gapped

$$\epsilon_{\pm\pm} = \pm \sqrt{\chi^2 + 4J_\pm^2 k^2},$$

but retains its birefringent property.⁴ The spectrum reduces to

that found with a usual mass term when $\beta = 0$.

B. Next Nearest neighbour interactions

In order to deal with next nearest neighbour interactions (with strength V_2) we again focus in the vicinity of a single Dirac cone, and make use of decompositions of $n_{AND} + n_{BNC}$ in Hartree and Fock channels (full details in Appendix A2). We hence introduce additional Hubbard Stratonovich fields to decompose this interaction term in the path integral. We find that for $V_1 = 0$, the leading instability is to the order parameter

$$\begin{aligned} \zeta_{35} &\propto \langle \bar{\psi} i\gamma_3 \gamma_5 \psi \rangle \\ &= -i \left[\langle \psi_A^\dagger \psi_D \rangle - \langle \psi_D^\dagger \psi_A \rangle + \langle \psi_B^\dagger \psi_C \rangle - \langle \psi_C^\dagger \psi_B \rangle \right], \end{aligned}$$

which breaks time reversal symmetry²⁹ and leads to an additional term in the action:

$$S_{\text{mix}}^{\text{nnn}} = \int_0^\beta d\tau \int d^2\mathbf{x} \zeta_{35} \bar{\psi} i\gamma_3 \gamma_5 \psi, \quad (10)$$

which has a critical coupling

$$\frac{1}{V_{35}} = \frac{2\Lambda}{\pi^2} \frac{1}{\beta^2(\beta^2 - 4)} \frac{1}{\gamma_+ + \gamma_-} \left\{ \frac{(1 - \beta^2 + \beta^2\gamma_+)}{2\sqrt{\gamma_+}} \ln \left| \frac{1 - \sqrt{\gamma_+}}{1 + \sqrt{\gamma_+}} \right| - \frac{(1 - \beta^2 - \beta^2\gamma_-)}{\sqrt{\gamma_-}} \tan^{-1} \left(\frac{1}{\sqrt{\gamma_-}} \right) \right\}, \quad (11)$$

where

$$\gamma_{\pm} = \pm \frac{(\beta^2 - 3)}{(\beta^2 - 4)} + \sqrt{\frac{(\beta^2 - 3)^2}{(\beta^2 - 4)^2} + \frac{(1 - \beta^2)^2}{\beta^2(4 - \beta^2)}}. \quad (12)$$

At a mean field level, this ordering is equivalent to having a circulating current in the pattern illustrated in Fig. 3, giving a quantum anomalous Hall phase similar to that found on the honeycomb lattice³³ or in the three band Hubbard model for cuprates³⁴ when next nearest neighbour interactions are present. If the order is as illustrated in Fig. 3, with an ampli-

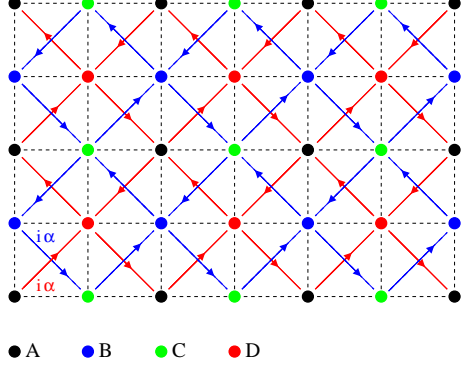


FIG. 3. Hopping pattern corresponding to non-zero ζ_{35} order parameter. The amplitude of all diagonal hopping integrals is identical, and equal to α .

tude α , then the dispersion is always gapped and rotationally invariant and takes the form

$$E_k = \pm \sqrt{2(J_+^2 + J_-^2)|\mathbf{k}|^2 + 16\alpha^2 \pm 2D_k} \quad (13)$$

$$D_k = \sqrt{(J_+^2 - J_-^2)^2|\mathbf{k}|^4 + 16(J_+^2 - J_-^2)\alpha^2|\mathbf{k}|^2}.$$

For non-zero α , there is always a gap of 8α between the upper and lower bands, but for non-zero β , the minimum gap is for a ring of finite k , with radius

$$k = \frac{2\sqrt{2}\alpha}{\sqrt{J_+^2 - J_-^2}} \left(\sqrt{1 + \frac{J_+^2 - J_-^2}{2J_-^2}} - 1 \right).$$

$$\frac{1}{V_{12}} = \frac{2\Lambda}{\pi^2} \frac{1}{\beta^2(\beta^2 - 4)} \frac{1}{\gamma_+ + \gamma_-} \left\{ \frac{(1 - \beta^2 + (2 - \beta^2)\gamma_-)}{\sqrt{\gamma_-}} \tan^{-1} \left(\frac{1}{\sqrt{\gamma_-}} \right) - \frac{(1 - \beta^2 - (2 - \beta^2)\gamma_+)}{2\sqrt{\gamma_+}} \ln \left| \frac{1 - \sqrt{\gamma_+}}{1 + \sqrt{\gamma_+}} \right| \right\}, \quad (14)$$

which tends to the same value as V_{35} in the limit $\beta \rightarrow 1$, at which H_0 and H_β have equal weight in the Hamiltonian.

The dispersion is shown as an insert in Fig. 4, which also illustrates the phase diagram when only next nearest neighbour interactions are present.

Now, as noted in Ref. 4, a different representation of the gamma matrices can transform $\gamma_0\gamma_1 \leftrightarrow \gamma_3$ and $\gamma_0\gamma_2 \leftrightarrow \gamma_5$, essentially swapping H_0 and H_β (up to a factor of β) in Eq. (3). The same transformation takes ζ_{35} to

$$\zeta_{12} \propto \langle \bar{\psi} i \gamma_1 \gamma_2 \psi \rangle$$

$$= i \left[\langle \psi_A^\dagger \psi_D \rangle - \langle \psi_D^\dagger \psi_A \rangle - \langle \psi_B^\dagger \psi_C \rangle + \langle \psi_C^\dagger \psi_B \rangle \right],$$

which also breaks time reversal symmetry and in which the direction of the circulating current on either the AD or BC

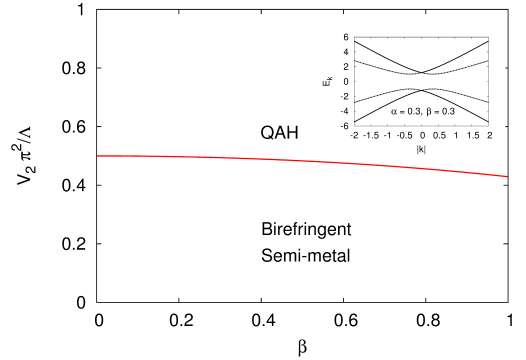


FIG. 4. Phase diagram as a function of β and V_2 for next nearest neighbour interactions, showing the birefringent semi-metal phase and the Quantum anomalous Hall (QAH) phase. The inset shows the dispersion for $\alpha = 0.3$ and $\beta = 0.3$.

sublattice is reversed with respect to ζ_{35} ordering. If $\beta > 1$ then ζ_{12} ordering is favoured and for $1 < \beta < 2$, the critical coupling is

C. Nearest neighbour and next nearest neighbour interactions

When both $V_1 \neq 0$ and $V_2 \neq 0$, there can be either χ or ζ_{35} ordering. We note that these two orders are even and odd

respectively under the discrete symmetry operator Γ that was introduced in Ref. 4, which in Euclidean form is

$$\Gamma = \frac{i}{2} (\gamma_2 \gamma_3 + \gamma_1 \gamma_5) - \frac{i}{2} (\gamma_1 \gamma_3 + \gamma_2 \gamma_5),$$

and on the lattice corresponds to a reflection about the diagonal AD in the unit cell, with $c_A \rightarrow c_A$, $c_B \rightarrow c_C$, $c_C \rightarrow c_B$ and $c_D \rightarrow -c_D$. The action of Γ on the Hamiltonian, H_k , is to exchange k_x and k_y . The effects on γ_0 , $\gamma_3 \gamma_5$, and $\gamma_1 \gamma_2$ are

$$\Gamma \gamma_0 \Gamma = \gamma_0, \quad \Gamma \gamma_3 \gamma_5 \Gamma = -\gamma_3 \gamma_5, \quad \Gamma \gamma_1 \gamma_2 \Gamma = -\gamma_1 \gamma_2.$$

We calculate the phase diagram for $V_1 \neq 0$ and $V_2 \neq 0$ below.

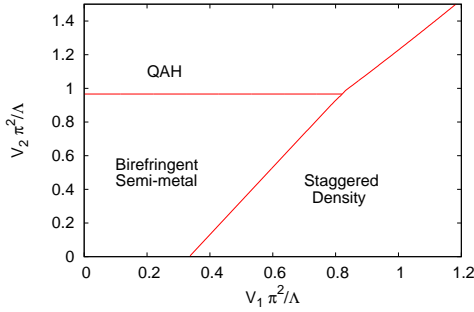


FIG. 5. Phase diagram as a function of V_1 and V_2 at $\beta = 0.5$

Note that the coupling strength at which the transition from semi-metal to QAH phase occurs differs between Figs. 4 and 5 because of a factor of 2 in the action. In calculating the phase diagram shown in Fig. 4 we assume that there are no other forms of ordering when performing the decomposition into Hubbard-Stratonovich fields, and in Fig. 5 we decompose the next-nearest neighbour interaction terms equally between χ and ζ_{35} ordering. The qualitative behaviour of the phase diagram shown in Fig. 5 is maintained for all β but the exact positions of the phase transition lines have some β dependence which can be anticipated from Figs. 2 and 4.

IV. LANDAU LEVELS

Whilst we originally derived birefringent fermions in a tight binding model with an artificial magnetic field,⁴ we are free to ask what the spectrum of birefringent fermions looks like when a magnetic field is present, without asking about specific tight binding models that might be required to realize them. Hence, we now derive the Landau level energy spectrum by coupling a magnetic field to the effective Hamiltonian. We write the Hamiltonian in real space, and use minimal coupling and the Landau gauge $\mathbf{A} = (0, Bx, 0)$. Noting that the system is translationally invariant in the y -direction we make the following ansatz for the eigenstates: $\psi^T(x, y) = e^{ik_y y} (f_1(x), f_2(x), f_3(x), f_4(x))$, and after the change of variables $\xi = (k_y + Bx)/\sqrt{B}$, $\epsilon = E/2J_0\sqrt{B}$ we obtain the eigenvalue equation

$$\begin{pmatrix} \epsilon & -(1+\beta)\xi & i(1+\beta)\frac{\partial}{\partial \xi} & 0 \\ (1+\beta)\xi & -\epsilon & 0 & i(1-\beta)\frac{\partial}{\partial \xi} \\ -i(1+\beta)\frac{\partial}{\partial \xi} & 0 & -\epsilon & (1-\beta)\xi \\ 0 & i(\beta-1)\frac{\partial}{\partial \xi} & (\beta-1)\xi & \epsilon \end{pmatrix} \begin{pmatrix} f_1(\xi) \\ f_2(\xi) \\ f_3(\xi) \\ f_4(\xi) \end{pmatrix} = 0. \quad (15)$$

We can combine the equations above to obtain the following differential equation for f_1 :

$$\left[\epsilon^2 - (1-\beta)^2 \xi^2 + (1-\beta)^2 \frac{\partial^2}{\partial \xi^2} \right] \left[\epsilon^2 - (1+\beta)^2 \xi^2 + (1+\beta)^2 \frac{\partial^2}{\partial \xi^2} \right] f_1(\xi) - (1-\beta^2)^2 f_1(\xi) = 0. \quad (16)$$

Using the ansatz $f_1(\xi) = H_n(\xi) \exp[-\xi^2/2]$, with $H_n(\xi)$ the n^{th} Hermite polynomial, we arrive at the following expression for the energy eigenvalues

$$\epsilon_{n,\beta,\pm\pm} = \pm \sqrt{(2n+1)(1+\beta^2)} \pm \sqrt{(2n+1)^2(1+\beta^2)^2 - 4n(n+1)(1-\beta^2)^2}.$$

To connect to the standard relativistic Landau levels, first consider $\beta = 0$, which gives $\epsilon_n = \pm\sqrt{2n+2}$, or $\epsilon_n = \pm\sqrt{2n}$. When $\beta = 0$, the system retains the $SU(2)$ chiral symmetry generated by $\{\gamma_3, \gamma_5, \gamma_3 \gamma_5\}$ and the Landau levels are doubly degenerate. When β is non-zero, the chiral symmetry is broken, and the degeneracy is lifted so that as $\beta \rightarrow 1$, half of the levels go to $\epsilon = 0$ and the other half to $\epsilon = \pm\sqrt{2(2n+1)}$ as illustrated in Fig. 6.

A. Integer Quantum Hall effect

In graphene the integer Quantum Hall effect shows plateaux at $\sigma = \pm(4n+2)e^2/h$, which is a result of the four fold de-

generacy (two from spin and two from valley degrees of free-

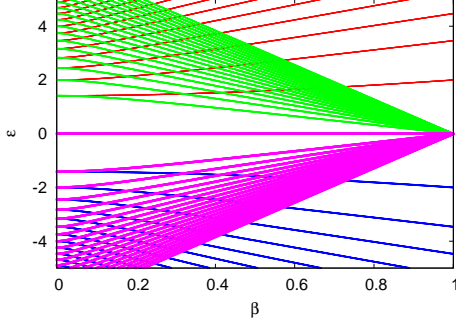


FIG. 6. Landau level energy eigenvalues as a function of β for $n = 0$ to $n = 20$.

dom) for $n \neq 0$ Landau levels and two fold degeneracy of the $n = 0$ Landau level.^{35,36} For spinless fermions as considered here, when $\beta = 0$ one would expect to see an integer quantum Hall effect with $\sigma = fe^2/h$ with $f = \pm(2n + 1)$, since there is no factor of 2 associated with spin degeneracy. When $\beta \neq 0$ the breaking of the degeneracy of the Dirac cones implies that the integer Quantum Hall effects will also be modified so that conductivities for all non zero integers should be present, i.e. $\sigma = fe^2/h$ with $f = \pm(n + 1)$.

V. BIREFRINGENT FERMIONS IN THREE DIMENSIONS

All of our discussions of birefringent fermions have focused on two dimensions, but it is interesting to ask whether this physics is realisable in three dimensions as well. Hosur *et al.* considered a staggered flux model in three dimensions, which has Dirac fermions as its low energy excitations. By choosing the same flux pattern and allowing for both J_+ and J_- hopping amplitudes in the 8 site unit cell as illustrated in Fig. 7, one can obtain birefringent fermions.

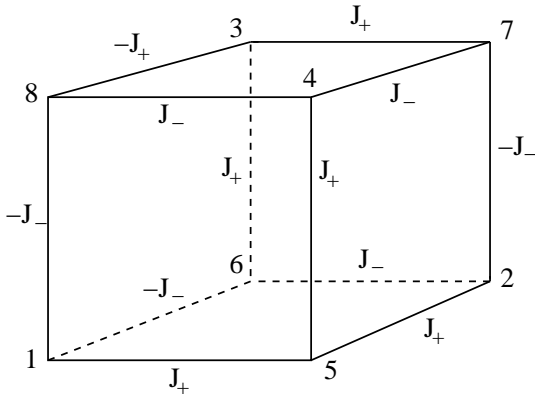


FIG. 7. Eight site unit cell and hopping parameters for the three dimensional birefringent fermion model.

Introduce an eight-component fermion operator at momentum \vec{k} : $f_{i\vec{k}}^\dagger = (A_{1\vec{k}}^\dagger, A_{2\vec{k}}^\dagger, A_{3\vec{k}}^\dagger, A_{4\vec{k}}^\dagger, A_{5\vec{k}}^\dagger, A_{6\vec{k}}^\dagger, A_{7\vec{k}}^\dagger, A_{8\vec{k}}^\dagger)$,

and then the tight binding Hamiltonian can be represented as

$$H = \sum_{\vec{k}} f_{i\vec{k}}^\dagger H_{ij\vec{k}} f_{j\vec{k}},$$

with the non-zero elements of the hopping matrix, H_k equal to:

$$\begin{aligned} H_{15} &= 2J_+ \cos k_x, & H_{16} &= -2J_- \cos k_y, & H_{18} &= \\ &-2J_- \cos k_z, & H_{25} &= 2J_+ \cos k_y, & H_{26} &= 2J_- \cos k_x, & H_{27} &= \\ &-2J_- \cos k_z, & H_{36} &= 2J_+ \cos k_z, & H_{37} &= 2J_+ \cos k_x, & H_{38} &= \\ &-2J_+ \cos k_y, & H_{45} &= 2J_+ \cos k_z, & H_{47} &= 2J_- \cos k_y, & H_{48} &= \\ &2J_- \cos k_x. \end{aligned}$$

This may be written in terms of the Pauli matrices in the following form

$$\begin{aligned} H_k &= 2J_0 [\cos k_x (\sigma_1 \otimes I_2 \otimes I_2) + \beta \cos k_x (\sigma_1 \otimes I_2 \otimes \sigma_3) \\ &+ \cos k_y (\sigma_2 \otimes I_2 \otimes \sigma_2) + \beta \cos k_y (\sigma_1 \otimes \sigma_3 \otimes \sigma_1) \\ &+ \cos k_z (\sigma_2 \otimes \sigma_2 \otimes \sigma_1) + \beta \cos k_z (\sigma_1 \otimes \sigma_1 \otimes \sigma_1)] \end{aligned}$$

Hence, the energy eigenvalues are given by

$$E_k = \pm 2J_\pm \sqrt{\cos^2 k_x + \cos^2 k_y + \cos^2 k_z},$$

and at the eight vertices of the Brillouin zone $\mathbf{K}_{\pm, \pm, \pm} = (\pm \frac{\pi}{2}, \pm \frac{\pi}{2}, \pm \frac{\pi}{2})$, there are Dirac points at which the dispersion takes the form $E_{\pm, \pm} = \pm 2J_\pm |k|$, and each of the birefringent fermion bands is doubly degenerate.

VI. DISCUSSION AND CONCLUSIONS

In conclusion, we have studied the effects of interactions within the tight binding model for birefringent fermions at a mean-field level. We have illustrated that for nearest neighbour and next-nearest neighbour repulsive interactions on the square lattice, the birefringent semi-metal persists up to a critical interaction strength for $\beta < 1$. We have calculated the β dependence of these critical interaction strengths and find that as β approaches 1, the system is more susceptible to interactions than for small values of β . A staggered density phase is favoured by nearest neighbour interactions and a quantum anomalous Hall phase with circulating currents is favoured by next-nearest neighbour interactions. The ordered phases that arise for birefringent fermions are thus quite similar to those of regular Dirac fermions, but as β is increased, birefringent fermions have a lower critical interaction strength for ordering. It should be noted that in the case of the honeycomb lattice, despite mean field predictions of quantum anomalous Hall phases,^{22,33,38} such phases have proven to be less robust in exact diagonalization calculations with periodic boundary conditions^{39,40}, but present when open boundary conditions are used.⁴¹ We expect that similar considerations apply to the situation considered here and consider the exploration of interaction effects beyond mean field theory to be an interesting avenue for future work.

Despite similarities at zero magnetic field, birefringent fermions display qualitatively different behaviour to regular Dirac fermions in the presence of a magnetic field. We considered the effects of a magnetic field on the spectrum of

birefringent fermions and obtained exact expressions for their Landau levels. The broken chiral symmetry of birefringent fermions lifts the degeneracy of Landau levels for regular Dirac fermions, and hence there are additional integer Quantum Hall plateaux compared to the case when $\beta = 0$.

The study presented here is part of the broader effort of understanding interaction effects in novel bandstructures. Future directions to consider include the addition of spin to birefringent fermions, which would also allow for on-site Hubbard interactions. The combination of interactions and magnetic field would also be interesting to investigate. The experimental realization of birefringent fermions would also be of great interest – we proposed a scheme to realize them in a cold atom

setting in Ref. 4 for cold atoms, but given the nature of the tight binding models for two and three dimensional birefringent fermions, it is conceivable that they might arise naturally in transition metal compounds where d -orbitals are important for hopping matrix elements.

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Appendix A: Useful Identities

For convenience, we list here the identities we used in decomposing interaction terms into Hartree and Fock channels.

1. Nearest Neighbour interactions

We can decompose nearest neighbour interaction terms into Hartree and Fock channels using the following identities: for the Hartree decomposition:

$$\begin{aligned} n_A n_B + n_A n_C + n_B n_D + n_C n_D &= \frac{1}{4} [(n_A + n_B + n_C + n_D)^2 - (n_A - n_B - n_C + n_D)^2] \\ &= \frac{1}{4} [(\bar{\psi}\gamma_0\psi)^2 - (\bar{\psi}\psi)^2] \end{aligned} \quad (A1)$$

and the Fock decomposition:

$$\begin{aligned} n_A n_B + n_A n_C + n_B n_D + n_C n_D &= -\frac{1}{8} \left\{ (\bar{\psi}\gamma_0\gamma_1\psi)^2 + (\bar{\psi}i\gamma_1\psi)^2 + (\bar{\psi}\gamma_0\gamma_3\psi)^2 + (\bar{\psi}i\gamma_3\psi)^2 \right. \\ &\quad \left. + (\bar{\psi}\gamma_0\gamma_2\psi)^2 + (\bar{\psi}i\gamma_2\psi)^2 + (\bar{\psi}\gamma_0\gamma_5\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right\}, \end{aligned} \quad (A2)$$

2. Next-Nearest Neighbour Interactions

The decompositions for next-nearest neighbour interactions are

$$\begin{aligned} n_A n_D + n_B n_C &= \frac{1}{8} [(n_A + n_B + n_C + n_D)^2 + (n_A - n_B - n_C + n_D)^2 \\ &\quad - (n_A - n_B + n_C - n_D)^2 - (n_A + n_B - n_C - n_D)^2] \\ &= \frac{1}{8} [(\bar{\psi}\gamma_0\psi)^2 + (\bar{\psi}\psi)^2 - (\bar{\psi}i\gamma_1\gamma_3\psi)^2 + (\bar{\psi}i\gamma_2\gamma_5\psi)^2], \end{aligned} \quad (A3)$$

for the Hartree channel and

$$n_A n_D + n_B n_C = -\frac{1}{8} [(\bar{\psi}i\gamma_1\gamma_5\psi)^2 + (\bar{\psi}i\gamma_2\gamma_3\psi)^2 + (\bar{\psi}i\gamma_3\gamma_5\psi)^2 + (\bar{\psi}i\gamma_1\gamma_2\psi)^2] \quad (A4)$$

for the Fock channel.

¹ K.S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov, *Science* **306**,

666 (2004); G. W. Semenoff, *Phys. Rev. Lett.* **53**, 2449 (1984).

- ² C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005); D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, Nature **452**, 970 (2008).
- ³ S. M. Young, S. Zaheer, J. C. Y. Teo, C. L. Kane, E. J. Mele, and A. M. Rappe, Phys. Rev. Lett. **108**, 140405 (2012).
- ⁴ M. P. Kennett, N. Komeilizadeh, K. Kaveh, and P. M. Smith, Phys. Rev. A **83**, 053636 (2011).
- ⁵ D. Bercioux, D. F. Urban, H. Grabert, and W. Häusler, Phys. Rev. A **80**, 063603 (2009).
- ⁶ R. Shen, L. B. Shao, B. Wang, and D. Y. Xing, Phys. Rev. B **81**, 041410(R) (2010).
- ⁷ V. Apaja, M. Hyrkäs, and M. Manninen, Phys. Rev. A **82**, 041402(R) (2010).
- ⁸ D. Green, L. Santos, and C. Chamon, Phys. Rev. B **82**, 075104 (2010).
- ⁹ N. Goldman, D. F. Urban, and D. Bercioux, Phys. Rev. A **83**, 063601 (2011).
- ¹⁰ Z. Lan, N. Goldman, A. Bermudez, W. Lu, P. Ohberg, Phys. Rev. B **84**, 165115 (2011).
- ¹¹ B. Dóra, J. Kailasvuori, and R. Moessner, Phys. Rev. B **84**, 195422 (2011).
- ¹² Chi-Ken Lu and Igor F Herbut, J. Phys. A: Math. Theor. **44**, 295003 (2011).
- ¹³ Z. Lan, A. Celi, W. Lu, P. Ohberg, and M. Lewenstein, Phys. Rev. Lett. **107**, 253001 (2011).
- ¹⁴ H. Watanabe, Y. Hatsugai, and H. Aoki, J. Phys.: Conf. Ser. **334**, 012044 (2011).
- ¹⁵ B. Roy, P. M. Smith, and M. P. Kennett, Phys. Rev. B **85**, 235119 (2012).
- ¹⁶ M. Vigh, L. Oroszlány, S. Vajna, P. San-Jose, G. Dávid, J. Cserti, and B. Dóra, Phys. Rev. B **88**, 161413(R) (2013).
- ¹⁷ L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, T. Esslinger, Nature **483**, 302 (2012).
- ¹⁸ K. K. Gomes, W. Mar, W. Ko, F. Guinea, and H. C. Manoharan, Nature **483**, 306 (2012).
- ¹⁹ M. Bellec, U. Kuhl, G. Montambaux, and F. Mortessagne, Phys. Rev. B **88**, 115437 (2013).
- ²⁰ I. F. Herbut, Phys. Rev. Lett. **97**, 146401 (2006).
- ²¹ V. A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993).
- ²² C. Weeks and M. Franz, Phys. Rev. B **85**, 041104(R) (2012).
- ²³ W.-F. Tsai, C. Feng, H. Yao, and J. Hu, arXiv:1112.5789v1.
- ²⁴ V. I. Iglovikov, F. Hébert, B. Grémaud, G. G. Batrouni, and R. T. Scalettar, arXiv:1404.5482v4.
- ²⁵ B. Seradjeh, C. Weeks, and M. Franz, Phys. Rev. B **77**, 033104 (2008).
- ²⁶ In Ref. 4 a Minkowski rather than Euclidean metric was used. This implies slightly modified expressions for $\gamma_{1,2,3,5}$.
- ²⁷ I. F. Herbut, Phys. Rev. B **83**, 245445 (2011).
- ²⁸ H. B. Nielsen and M. Ninomiya, Nucl. Phys. **185**, 20 (1981).
- ²⁹ F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
- ³⁰ E. Dagotto, E. Fradkin, and A. Moreo, Phys. Lett. **172**, 383 (1986).
- ³¹ I. F. Herbut, V. Juričić, and B. Roy, Phys. Rev. B **79**, 085116 (2009).
- ³² K. Kaveh and I. F. Herbut, Phys. Rev. B **71**, 184519 (2005).
- ³³ S. Raghu, X.-L. Qi, C. Honerkamp, and S.-C. Zhang, Phys. Rev. Lett. **100**, 156401 (2008).
- ³⁴ C. Weber, A. Läuchli, F. Mila, and T. Giamarchi, Phys. Rev. Lett. **102**, 017005 (2009).
- ³⁵ K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Nature **438**, 197 (2005).
- ³⁶ V. P. Gusynin and S. G. Sharapov, Phys. Rev. Lett. **95**, 146801 (2005).
- ³⁷ P. Hosur, S. Ryu, and A. Vishwanath, Phys. Rev. B **81**, 045120 (2010).
- ³⁸ A. G. Grushin, E. V. Castro, A. Cortijo, F. de Juan, M. A. H. Vozmediano, and B. Valenzuela, Phys. Rev. B **87**, 085136 (2013).
- ³⁹ M. Daghofer and M. Hohenadler, Phys. Rev. B **89**, 035103 (2014).
- ⁴⁰ N. A. García-Martínez, A. G. Grushin, T. Neupert, B. Valenzuela, and E. V. Castro, Phys. Rev. B **88**, 245123 (2013).
- ⁴¹ T. Durić, N. Chancellor, and I. F. Herbut, Phys. Rev. B **89**, 165123 (2014).